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ON THE REDUCTION OF DIFFERENTIAL EQUATIONS TO THE NORMAL FORM BY AN ANALYTIC TRANSFORMATION

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We treat the problem of reducing a system of nth-order ordinary differential equations to normal form in a neighborhood of a singular point in the presence or absence of resonances. We have shown that such a reduction is possible in the class of analytic transformations if the original system admits of an analytic symmetry group of specified dimension.

We examine the real nth-order autonomous system

$$x^{\cdot} = f(x), \qquad f(0) = 0 \tag{1}$$

We assume that the vector-valued function $f(x)$ is analytic in a neighborhood of the point $x = 0$, that among the eigenvalues λ_k of the linear part there are no multiple ones, and that only a finite number m of linearly independent formal operators

$$X = \sum \xi_i(x)\, \partial / \partial x_i \not\equiv \mu L$$

exist ($\xi_i(x)$ are formal power series), commuting with the shift operator

$$L = \Sigma f_i(x) \partial / \partial x_i$$

along the trajectories of system (1), $[L, X] = 0$. With system (1) we associate a finite-dimensional maximal group G of analytic transformations of a neighborhood of point $x = 0$ preserving this system, namely, an analytic symmetry group (cf. [1]) (the elements of algebra $\mathbf{L}$ of group G are infinitesimal analytic operators). Let l be the number of independent resonance relations $\lambda_1 k_1 + \ldots + \lambda_n k_n = 0$ $(0 \leqslant l \leqslant n-1,\ k_i \geqslant 0)$.

Theorem. For system (1) to be reducible to normal form in a neighborhood of point $x = 0$ by an analytic transformaltion, it is sufficient, and for $l = 0, 1$ also necessary, to fulfil the condition $\dim G = m$.

Proof. Let φ be an invertible transformation reducing system (1) to the normal form

$$y_i^{\cdot} = y_i p_i, \qquad i \leqslant n \tag{2}$$

It is easy to verify that system (2) admits of a group with the operators

$$Y = \sum_{i=1}^{n_*} \alpha_i y_i \frac{\partial}{\partial y_i} \tag{3}$$

in which for $l \neq 0$ the constants α_i are related by independent relations of the form

$$\alpha_1 k_1 + \ldots + \alpha_n k_n = 0 \tag{4}$$

where the integers $k_i \geqslant 0$, $\Sigma k_i > 0$. For $l = 0$ only $\alpha_i \neq \mu\lambda_i$ is necessary.

Sufficiency. Let $\dim G = m$. Then all the operators $X \in \mathrm{L}$ are analytic. Consequently, the corresponding Lie equations

$$dx_i' \,/\, d\tau = \xi_i\,(x') \tag{5}$$

where τ is the canonic group parameter, are analytic. Further, constants $c_1, \ldots, c_{m+1}$ can be found such that an Eq. (5) corresponding to the operator

$$X = c_1 X_1 + \ldots + c_m X_m + c_{m+1} L$$

is reduced by transformation φ to the form

$$dy_i' = \alpha_i y_i' \tag{6}$$

Here on the numbers α_i we can impose, in addition to conditions (4), $n - l - 1$ more conditions of the same form. We obtain identically a resonance system (6) not having small denominators. But then, according to [2], the transformation φ is analytic.

Necessity. Let transformation φ be analytic. If $l = 0$, then $p_i \equiv \lambda_i$ in Eqs. (2) [3, 4]. In this case system (2) does not admit of analytic first integrals. Therefore, the system of defining equations

$$\sum_{i=1} \lambda_i y_i \frac{\partial \eta_j}{\partial y_i} = \lambda_j \eta_j$$

has only those analytic solutions which correspond to operator (3). For $l = 1$ the functions p_i contain a finite number of terms, while the base of the algebra corresponding to the symmetry group of system (2) is formed by only (commuting) operators (3) [5]. Thus, system (2) is analytic for $l = 0, 1$ and $m = n - 1$.

The transformation φ^{-1} converts each operator Y into some operator X commuting with L. If φ is a convergent transformation, then φ^{-1} also converges. Since all operators Y are analytic, the corresponding operators X are necessarily analytic. By Lie's theorem, to algebra $\boldsymbol{L}$ there corresponds an analytic symmetry group of the same dimension: $\dim G = m$. The theorem is proved.

Several obvious corollaries follow from it.

Corollary 1. For the local analytic equivalence of two systems of form (1) with groups G and G' it is necessary:

a) that the systems be formally equivalent, i.e. that their finite-dimensional invariants coincide (cf. [6]);

b) that the groups G and G' be locally isomorphic; when conditions (a), (b) are fulfilled, it is sufficient that $\dim G = m$.

Corollary 2. When $l = 0, 1$ system (1) is reduced to normal form by an analytic transformation if and only if group G is commutative and $\dim G = n - 1$.

Corollary 3. When $l = 1$ all the analytic normal forms obtained from the given system (1) by formal transformations are analytically equivalent.

Notes. 1) The question on the reduction of differential equation systems to normal form has been investigated in many papers, most completely in [4]. The group-theoretic criterion formulated in the present note is, from the author's point of view, of a certain fundamental interest and has the practical advantage that the problem of looking for the analytic algebra L is linear. In addition, if the Zigel condition $|\Sigma k_i\lambda_i - \lambda_j| > ak^{-\nu}$ [3] is fulfilled when $l = 0$, then all the singular integral surfaces and curves of system (1) are described by analytic equations (in components of operators X) within a region in which the operators X can be analytically continued from the neighborhood of point $x = 0$ wherein they are analytic by virtue of the theorem proved. These integral manifolds can be found approximately.

2) In this paper we have examined only systems with a finite-dimensional algebra of operators. Such systems cannot automatically admit of formal first integrals and in this sense are nonsingular. Therefore, real systems with an analytic symmetry group prove to be outside the scope of Theorem III of [4].

3) Although the situation is somewhat different in Hamiltonian systems, the theorem proved in this paper indicates a definite resemblance with the situation established by Rüssman results in [7], according to which the existence of an additional analytic integral guarantees the convergence of the Birkhoff transformation (to such an integral in a system with two degrees of freedom there corresponds a one-parameter analytic symmetry group).

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